PHYSICAL REVIEW E

STATISTICAL PHYSICS, PLASMAS, FLUIDS, AND RELATED INTERDISCIPLINARY TOPICS

RAPID COMMUNICATIONS

The Rapid Communications section is intended for the accelerated publication of important new results. Since manuscripts submitted to this section are given priority treatment both in the editorial office and in production, authors should explain in their submittal letter why the work justifies this special handling. A Rapid Communication should be no longer than 4 printed pages and must be accompanied by an abstract. Page proofs are sent to authors.

Exact solution of a two-dimensional random Ising model

Maurizio Serva

Dipartimento di Matematica, Università dell'Aquila and INFM, I-67010 Coppito, L'Aquila, Italy
(Received 22 May 1997)

The model considered is a $d=2$ layered random Ising system on a square lattice with a nearest-neighbor interaction. It is assumed that all the vertical couplings are equal and take the positive value J, while the horizontal couplings are quenched random variables that are equal in the same row but can take the two possible values J and $J-K$ in different rows. The exact solution is obtained in the limit case $K \to \infty$ for any distribution of the horizontal couplings. The model that corresponds to this limit can be seen as an ordinary Ising system where the spins of some rows, chosen at random, are frozen in an antiferromagnetic order. No phase transition is found if the horizontal couplings are independent random variables, while for correlated disorder one finds a low temperature phase with some glassy properties. [S1063-651X(97)50309-6]

PACS number(s): 05.50.+q, 02.50.−r

Ising spin glasses have been solved exactly in their mean field version [1,2], while as far as I know no exact solutions are available at finite dimensionality $d \geq 2$. Indeed, in the presence of disorder even a $d=1$ system with a magnetic field is a very complicated problem [3] and compact exact solutions can be found only in special cases [4]. This is frustrating, since it is not always clear if the qualitative results of a mean field approximation are shared by the finite dimension model. In this paper I am far from answering this problem; nevertheless, I exactly solve a class of $d=2$ layered random Ising systems that in some conditions have a low temperature phase. The nature of this nonferromagnetic low temperature phase is still unclear to me, but there are some indications that it shares some of the properties of a glassy phase.

The models I consider are defined as follows: the interaction is effective only between nearest neighbors on a square lattice; all the vertical couplings are equal, while the horizontal couplings are quenched variables that are equal in the same row but can take different values in different rows; the vertical couplings take the positive value J, while the horizontal couplings J_i can take the two possible values J and $J-K$, with $K \to \infty$. These models have frustration since the product of the signs of the coupling around a plaquette

can be negative. Different distributions of the horizontal couplings correspond to different models of the class; in the simplest case, the J_i are independent random variables and take the value J with probability $1-p$ and $J-K$ with probability p.

Layered Ising models of this type had been first considered by McCoy and Wu [5,6] for the nonfrustrated case, which is used for studying the effect of quenched randomness on the ferromagnetic-paramagnetic transition. These authors dealt with the determinant that occurs in the Pfaffian approach, and while they do not provide an explicit exact solution of the problem they are able to show that the free energy has an infinitely differentiable singularity at the transition. Layered models with frustration have been studied by Shankar and Murthy [7]; not only their topic but their approach is similar to this work, since they deal with the row to row transfer matrices. They do not find out an exact solution; nevertheless, they map the problem into a collection of $d=1$ random field Ising systems from which they can extract a lot of information. In particular, they provide evidence for the existence of a low temperature phase.

Let me now state the problem more precisely. Assuming that $N=LM$ is the number of spins, L is the number of rows, and M the number of columns, the Hamiltonian can be written as

MAURIZIO SERVA

$$H_N = -\sum_{ij} (J\sigma_{i,j}\sigma_{i+1,j} + J_i\sigma_{i,j}\sigma_{i,j+1}), \tag{1}$$

where the J_i are the horizontal couplings whose values only depend on the row i and not on the column j. One can write $J_i = J - \eta_i K$, where the quenched variables η_i can take the values 0 and 1 according to a given distribution. In the independent case $\eta_i = 0$ with probability $1-p$ and $\eta_i = 1$ with probability p. The partition function is

$$Z_N = \sum_{\{\sigma\}} \exp\left\{\sum_{ij} \beta[J\sigma_{i,j}\sigma_{i+1,j} + J\sigma_{i,j}\sigma_{i,j+1} - \eta_i K(1 + \sigma_{i,j}\sigma_{i,j+1})]\right\}, \tag{2}$$

where the constant term $\Sigma_{ij}\eta_i K$ has been added to the Hamiltonian in order to avoid divergences in the $K \to \infty$ limit. After having defined $\Gamma \equiv J\beta$ and performed the limit $K \to \infty$ one obtains

$$Z_N = \sum_{\{\sigma\}} \prod_{ij} \left[\exp\{\Gamma\sigma_{i,j}\sigma_{i+1,j} + \Gamma\sigma_{i,j}\sigma_{i,j+1}\} \times \left(1 - \frac{1 + \sigma_{i,j}\sigma_{i,j+1}}{2}\eta_i\right)\right]. \tag{3}$$

The terms in parentheses equal 1 when $\eta_i = 0$ and $(1 - \sigma_{i,j}\sigma_{i,j+1})/2$ when $\eta_i = 1$. Notice that in this second case the antiferromagnetic order between neighbor spins on the row is imposed; in fact, if $\sigma_{i,j}$ and $\sigma_{i,j+1}$ have the same sign they give a vanishing contribution to the partition function. It is now clear that Eq. (3) defines a class of Ising model with both vertical and horizontal couplings equal to J and with the spins of some rows frozen in an antiferromagnetic order. The frustration comes out from the fact that the tendency to the ferromagnetic alignment due to the positive couplings is in competition with the tendency to the antiferromagnetic alignment induced by the frozen spins on the unfrozen ones. A similar problem, where the spins are randomly frozen in a random direction, has been solved in $d = 1$ in [4] and studied in $d = 2$ at zero temperature in [8].

The advantage of considering layered disorder is that one can apply a standard diagonalization method [7,9], and reduce the problem to the evaluation of the trace of products of random matrices. Following the same steps of [7,9] one easily finds the free energy

$$f = -\frac{J}{2\Gamma} \log(2 \sinh 2\Gamma) - \frac{J}{2\pi\Gamma} \int_0^\pi \gamma(q,\Gamma)dq, \tag{4}$$

where

$$\gamma(q,\Gamma) = \lim_{L\to\infty} \frac{1}{L} \log \text{Tr} \prod_{i=1}^{L} T_i(q,\Gamma). \tag{5}$$

The 2×2 matrices $T_i(q,\Gamma)$ can be written as the product $T_i(q,\Gamma) = E_i T(q,\Gamma)$ where the

$$E_i = \begin{pmatrix} 1 & 0 \\ 0 & 1-\eta_i \end{pmatrix} \tag{6}$$

are random and equal the identity when $\eta_i = 0$ and the up projector $\tau^+ = (1 + \tau_3)/2$ when $\eta_i = 1$. The matrix $T(q,\Gamma)$, on the contrary, is constant and reads

$$T(q,\Gamma) = \exp\{-\Gamma\tau_3\}\exp\{2\Gamma^*(\tau_3 \cos q + \tau_1 \sin q)\} \times \exp\{-\Gamma\tau_3\}, \tag{7}$$

where τ_1 and τ_3 are Pauli matrices and $\Gamma^* \equiv -\frac{1}{2}\log(\tanh\Gamma)$.

The trace of a product of random matrices is easily accessible via computer simulation but it cannot be, in general, exactly computed. In the present case, nevertheless, following a similar method as in [4], it is possible to find out the compact analytical result. Consider a given realization of the quenched variables η_i and look at the product of matrices in Eq. (5). Since $T_i(q,\Gamma) = E_i T(q,\Gamma)$ that product reduces to a product of matrices $T(q,\Gamma)$ and up projectors τ^+. The first and second τ^+ will be separated by l_1 matrices $T(q,\Gamma)$, the second and third by l_2 matrices $T(q,\Gamma)$, and so on. The l_n are random variables that can take the values 1,2,.... whose distribution can be easily found out once the distribution of the η_i is given ($l_n - 1$ is the number of unfrozen rows between two frozen ones). The order number n goes from 1 to $n_f = L/\bar{l}$; in fact, one must have $\Sigma_{n=1}^{n_f} l_n = L$ so that $\Sigma_{n=1}^{n_f} l_n/n_f \equiv \bar{l} = L/n_f$. With the help of these considerations one can rewrite Eq. (5) as

$$\gamma(q,\Gamma) = \lim_{L\to\infty} \frac{1}{L} \log \prod_{n=1}^{L/\bar{l}} [T(q,\Gamma)^{l_n}]_{11}$$

$$= \lim_{L\to\infty} \frac{1}{L} \sum_{n=1}^{L/\bar{l}} \log[T(q,\Gamma)^{l_n}]_{11}, \tag{8}$$

where $[T(q,\Gamma)^{l_n}]_{11}$ is the upper left entry of $T(q,\Gamma)^{l_n}$. If $P(l)$ is the probability that two successive rows of infinitely negative couplings are separated by $l-1$ rows of finite positive couplings, then $\bar{l} \equiv \Sigma_{l=1}^{\infty} lP(l)$ and

$$\gamma(q,\Gamma) = \sum_{l=1}^{\infty} \frac{1}{\bar{l}} P(l)\log[T(q,\Gamma)^l]_{11}. \tag{9}$$

In order to find the explicit form for Eq. (9) it is convenient to write

$$T(q,\Gamma) = \exp\{\epsilon(\tau_3 \cos\phi + \tau_1 \sin\phi)\}, \tag{10}$$

where

$$\cosh\epsilon = \frac{\cosh^2 2\Gamma}{\sinh 2\Gamma} - \cos q \tag{11}$$

and

$$\cos\phi = \frac{(\cosh 2\Gamma)(\cos q - \sinh 2\Gamma)}{(\sin^2 q + (\cosh^2 2\Gamma)(\cos q - \sinh 2\Gamma)^2)^{1/2}}. \tag{12}$$

Using Eq. (10) we directly obtain

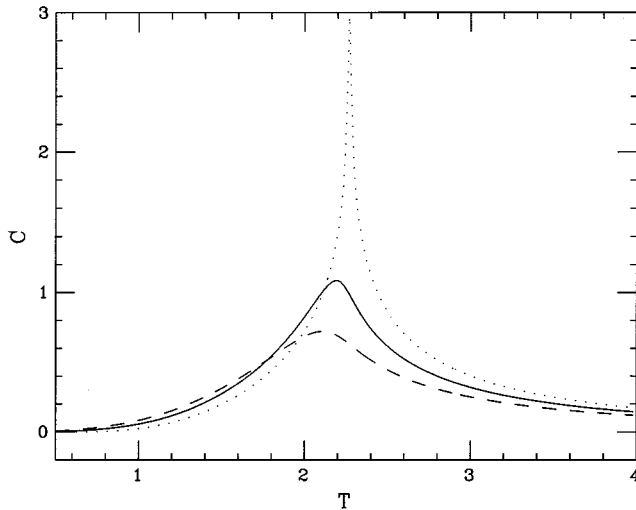$$[T(q,\Gamma)^l]_{11} = \cosh(l\epsilon) + \cos\phi \sinh(l\epsilon), \tag{13}$$

FIG. 1. Specific heat C as a function of the temperature T. The dotted line corresponds to the Ising model ($p=0$), the full line to $p=0.1$, and the dashed line to $p=0.2$.

Finally,

$$f = -\frac{J}{2\Gamma} \log(2 \sinh 2\Gamma) - \frac{J}{2\pi\Gamma\bar{l}} \sum_{l=1}^{\infty} P(l) \int_0^\pi \log(\cosh(l\epsilon)$$
$$+ \cos \phi \, \sinh(l\epsilon))dq, \qquad (14)$$

where ϵ and ϕ are given in Eqs. (11) and (12).

The probability $P(l)$ for the simplest choice of independent η_i is $P(l) = p(1-p)^{l-1}$ and $\bar{l} = 1/p$. In this case one can prove that the system has no phase transition, except for $p=0$, where it trivially reduces to the ordinary Ising model. In Fig. 1 the specific heat C is shown in correspondence with different values of p; one can notice that the logarithmic divergence is smoothed, showing the absence of transition. Nevertheless, the model is frustrated and its zero temperature properties are not completely trivial. One can compute the $T=0$ energy f_0 and entropy s_0 and finds
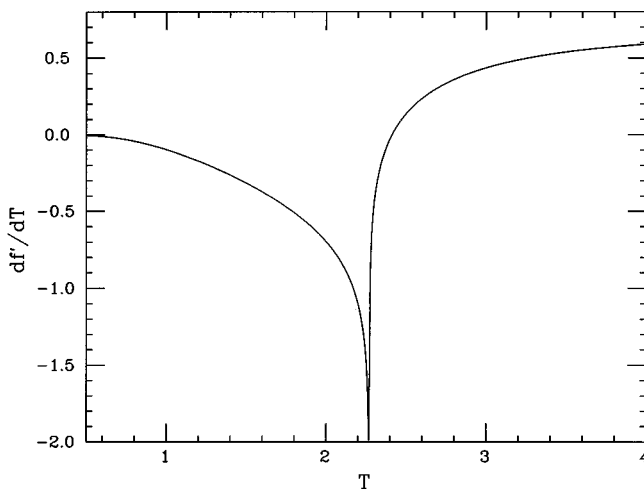
$$f_0 = -2J(1-p)^2, \qquad (15)$$

FIG. 2. Temperature derivative df'/dT of $f' \equiv [\partial f/\partial p]_{p=0}$ as a function of T_s.
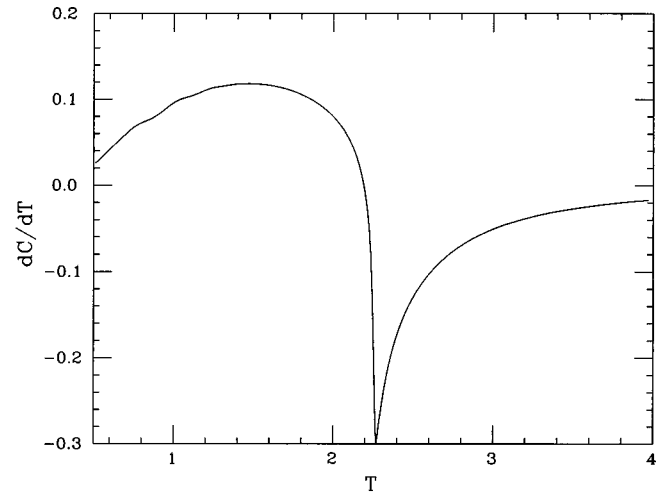

FIG. 3. Temperature derivative dC/dT of the specific heat for the $P(l) = a/l^3$ model.

$$s_0 = Jp^2(1-p)\log\left(\frac{\sqrt{5}+1}{2}\right). \qquad (16)$$

s_0 is not vanishing for $p \neq 0,1$ showing an exponential degeneration of the ground state due to the frustration of the model.

Since the transition disappears for $p \neq 0$ the role of p reminds one of a magnetic field, which also suppresses the transition. The analog of the spontaneous magnetization is obtained in the limit $p \to 0$ as

$$f' \equiv \left[\frac{\partial f}{\partial p}\right]_{p=0} = -\frac{J}{2\pi\Gamma} \int_0^\pi \log\left(\frac{1+\cos \phi}{2}\right)dq. \quad (17)$$

This quantity is continuous while its derivative df'/dT is not as shown in Fig. 2, where one can see a logarithmic divergence at T_c (the Onsager critical temperature).

The circumstance that a phase transition can be found only at $p=0$ suggests a more careful look at the model around this value. If one chooses $p = \alpha/L$ one has a vanishing p in the thermodynamic limit and the free energy is the same as that of the standard Ising model. Nevertheless, one has a random finite number of frozen rows. This number is Poisson distributed with intensity α and it is different for different realizations of the disorder (no self-averaging). The distance between two given frozen rows is also a random number of order of L and it also varies from one realization to another. The final result is that the frozen rows separate a random number of regions of random size of order N whose magnetization at $T \leq T_c$ is $\pm m(T)$ independently one from the other [$m(T)$ is the Onsager spontaneous magnetization at temperature T]. As a consequence of this fact, the whole system can be in all the states corresponding to all the possible combinations of magnetization of each region. In conclusion, one has the same free energy of a standard Ising system but a number of pure states each of them corresponding to a different local magnetization. The situation is completely analogous to that studied in [10] for a diluted $d=1$

model at zero temperature. Following the same line of [10] it is easy to compute the overlap probability; in particular, for large α one has

$$P(q) \simeq \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{q^2}{2t}\right\}, \tag{18}$$

where $t = m(T)^2/\alpha$. Equation (18) implies that the overlap between two different pure states vanishes in the limit $\alpha \to \infty$; nevertheless, it should be noticed that the self-overlap $q_{max} = m(T)^2$, being independent on α, remains finite.

To summarize, for $p \neq 0$ there is no phase transition, while for $p = \alpha/L$ one has a glassylike phase that comes out from an artificial construction that maintains the same free energy of the Ising model. It is straightforward, at this point, to look at an intermediate situation, where the number of frozen rows is of order of L but they can be much more separated than in the independent case. This task can be accomplished with the choice $P(l) = a/l^3$ (a is the normalization constant) which replaces the exponential distribution $P(l) = p(1-p)^{l-1}$ of the independent case. By substituting this expression in Eq. (14) one can easily compute the free energy and look at the eventual divergences. In spite of the fact that the free energy is now different from that of the standard Ising model, one still finds a phase transition at the Onsager temperature. Nevertheless, this phase transition does not correspond to a divergence in the specific heat, but in its derivative dC/dT, which is plotted in Fig. 3. I have not been able to quantitatively characterize the low phase temperature with an order parameter. In this phase, in fact, while the spontaneous magnetization vanishes, both the overlap and a parameter connected with the antiferromagnetic order seems to differ from zero. I hope that some light on this point will come out from future research.

[1] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **32**, 1792 (1975).

[2] M. Mezard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1988).

[3] G. Paladin and M. Serva, Phys. Rev. Lett. **69**, 706 (1992).

[4] G. Grinstein and D. Mukamel, Phys. Rev. B **27**, 4503 (1983).

[5] B. M. McCoy and T. T. Wu, Phys. Rev. **76**, 631 (1968).

[6] B. M. McCoy, Phys. Rev. B **2**, 2795 (1970).

[7] R. Shankar and Ganpathy Murthy, Phys. Rev. B **36**, 536 (1987).

[8] B. Derrida and Y. Shnidman, J. Phys. (France) Lett. **45**, L507 (1984).

[9] T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. **36**, 856 (1964).

[10] A. Crisanti, G. Paladin, M. Serva, and A. Vulpiani, J. Phys. I **3**, 1993 (1993).